Indian Statistical Institute, Bangalore
B. Math.(Hons.) I Year, First Semester
Semesteral Examination
Analysis -I

Time: 3 hours   November 27, 2009   Instructor: Pl.Muthuramalingam

The Paper has 50 marks.   Max marks you can get is 40

1. Let y be any irrational number in $[0, 1]$. Let $\frac{p_n}{q_n} \longrightarrow y$, p_n, q_n are positive integers with $gcd\{p_n, q_n\} = 1$. Show that $q_n \longrightarrow \infty$. [3]

2. Let $f : [0, 1] \longrightarrow [0, \infty)$ be any function assume that there exists $M \geq 0$ such that for all subsets $\{x_1, x_2, \cdots, x_k\}$ of $[0, 1]$, one has
$f(x_1) + f(x_2) + \cdots + f(x_k) \leq M$. Show that
$G = \{x : f(x) \neq 0\}$ is a countable set. [2]

3. Let f be a bounded increasing function on $(0, 1)$. Let x_0 be in $(0, 1)$. Show that the left limit for f at x_0 viz $\lim\limits_{x \longrightarrow x_0} \{f(x) : x < x_0\}$ exists and $= \sup\limits_{x<x_0} f(x)$. [2]

4. Let $g : [0, 1] \longrightarrow R$ be any continous function with $g(0) < 0 < g(1)$. Show that g assumes the value 0. [3]

5. Let $f : R \longrightarrow R$ be any infinitely differentiable function. State and prove Taylor's theorem on the interval $[x_0, x_0+h]$ involving $f, f', f^{(2)}, \cdots,$ $f^{(n+1)}$ for any $n \geq 1$. [5]

6. Let $f : [a, b] \longrightarrow R$ be continuous, differentiable and f' be continuous. Show that $\lim\limits_{\delta \longrightarrow 0} \sup\limits_{a \,\leq\, x \,\leq\, b} \sup\limits_{0 \,<|t-x|\, \leq\, \delta} \left| \frac{f(t)-f(x)}{t-x} - f'(x) \right| = 0$ [2]

7. Let $a_1, a_2, \cdots$ be a sequence of reals converging to 0. Let $\alpha : \{1, 2, 3, \cdots\} \longrightarrow \{1, 2, 3, \cdots\}$ be any 1-1 and onto map. Put $b_j = a_{\alpha(j)}$. Show that $b_j \longrightarrow 0$. [2]

8. Let a and b be real numbers. If the series $(a+b)+(a+2b)+(a+3b)+\cdots$ is convergent, then show that $b = 0$ and $a = 0$. [2]

9. Let A, B be bounded subsets of $[0, \infty)$. Let $C = \{ab : a \in A, b \in B\}$. Let $x = \sup A, y = \sup B, z = \sup C$. Note that x need not be in A and y need not be in B. Show that $z = xy$. [2]

10. (a) Let $a_1, a_2, \cdots$ be a sequence with $a_j \geq 0$. Let $\sum\limits_1^\infty a_j$ be convergent let $n_1 < n_2 < n_3 < \cdots$ be increasing sequence of natural numbers let $b_j = a_{n_j} \cdots$. Show that $\sum b_j$ is convergent. [2]

(b) Give an example of a real sequence $x_1, x_2, \cdots$ and a subsequence $x_{n_2}, x_{n_2} \cdots$ such that $\sum x_j$ is convergent and $\sum x_{n_j}$ is not convergent prove your claim.

11. Let $a_j > 0$ for $j = 1, 2, 3, \cdots$. Assume that $\frac{a_{j+1}}{aj}$ and $a_j^{1/j}$ are also bounded sequences. Show that

$\limsup\limits_{j \longrightarrow \infty} a_j^{1/j} \leq \limsup\limits_{j \longrightarrow \infty} a_{j+1}/a_j.$ [3]

12. Define $f : [0,1] \longrightarrow R$ by $f(x) = x^a \sin(\frac{1}{x^c})$, where $c > o$ and $a \geq 0$ for $x > 0$, $f(0) = 0$.

(i) If f is continuous at 0, show that $a > 0$. [2]

(ii) If f is differentiable at 0, show that $a > 1$. [1]

13. Let $f : [0, \infty) \longrightarrow R$ be continuous and differentiable on $(0, \infty)$. Further let $f(0) = 0$ and f' be increasing on $(0, \infty)$. Define $g(x) = \frac{f(x)}{x}$ for $x > 0$. Show that g is increasing on $(0, \infty)$. [3]

14. Let $a_{ij} : R \longrightarrow R$ be c^∞ functions for $i, j = 1, 2, 3$. Let

$\alpha(x) = \det\ [(a_{ij}(x))]_{i,j}$.

Show that the derivative of α can be written as a sum of three determinants involving a_{ij} and its derivatives. Be as explicit as possible.

[4]

15. Let $f : R \longrightarrow R$ be any C^2 function *i.e.*, f, f', f'' are all continuous. Assume that $M_0 = \sup\limits_x \mid f(x) \mid$ and $M_2 = \sup\limits_x \mid f''(x) \mid$ are both finite. Show that f' is a bounded function. Put $M_1 = \sup\limits_x \mid f'(x) \mid$. Show that $M_1^2 \leq 4M_0\ M_2$.

[4]

16. Let $a_1, a_2, \cdots$ be a sequence of reals with $\sum a_j$ convergent. Let $n_1 < n_2 < n_3 < \cdots$.

Put $b_1 = a_1 + a_2 + \cdots + a_{n_1}$,

$b_2 = a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}$.

$b_3 = a_{n_2+1} + a_{n_2+2} + \cdots + a_{n_3}, etc.$

Show that the series $b_1 + b_2 + \cdots$ is convergent and converges to $\sum a_r$.

[2]

17. Show that any disjoint collection of bounded intervals each of positive length is finite or countable.

[2]

18. Let $f : (a, b) \longrightarrow R$ be continuous every where and differentiable at x_0 in (a, b). Let $x_0 < a_n < b_n$ and $b_n \longrightarrow x_0$, if $\frac{b_n - x_0}{b_n - a_n}$ is a bounded sequence, show that $\frac{f(b_n) - f(a_n)}{b_n - a_n} \longrightarrow f'(x_0)$. [2]